

Poincare Cartan Form for Gauge Fields in Curved Background.

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Abstract

The ‘directly Hamiltonian’ field theory in the extended phase space is applied to gauge fields in curved spacetime background. These fields being differential 1-forms, have canonical momenta which are 2-forms. The Poincare-Cartan 4-forms for matter and gauge fields have to be modified with the exterior derivatives replaced by the covariant derivative for maintaining gauge invariance.

1 Introduction

In a recent communication [1] (called I here) a Hamiltonian formalism is developed in which scalar fields are associated with canonical momenta which are differential 1-forms. This is a ‘directly Hamiltonian’ formalism where canonical momenta are not defined through a Lagrangian via time derivatives of fields. Rather, manifest invariance in arbitrary curved spacetime is maintained at each step. The main features of this formalism are as follows. For background, motivation and other details see I.

1. If we treat all the four derivatives $\partial_\mu \phi$ of the field ϕ as one quantity then it follows we should allow four components in momentum $p = p_\mu dx^\mu$ to be associated with one field variable ϕ . Thus canonical momentum is a differential 1-form for a field which is a form of degree zero. The ‘coordinate’ ϕ and its ‘canonical

momentum' p are both differential forms but with the momenta being forms of a degree higher.

2. The central object in Hamiltonian mechanics is the Poincare-Cartan (PC) differential 1-form $pdq - Hdt$. The PC-form in field theory is a differential 4-form constructed in the extended phase space. The standard structure of PC-form has a 'fundamental' part pdq which is linear in the differential of the coordinates. This part determines the relation of canonical momentum to gradient of the field. The remaining part of PC-form is the Hamiltonian which is constructed out of fields and momenta. For a single free scalar field, for example, we can write

$$\Theta = (*p) \wedge d\phi - H, \quad H = \frac{1}{2}(*p) \wedge p + \frac{1}{2}m^2\phi^2(*1)$$

where the Hodge dual star operator associated with the Riemannian metric $g_{\mu\nu}$ of spacetime is used to convert 1-form p into 3-form $*p$ and 0-form (here constant 1) into volume 4-form $(*1)$. See the appendix A for notation.

3. In general the Hamiltonian H is constructed as a 4-form (which is equivalent to a scalar) with ϕ and p just like a Lagrangian is constructed from ϕ and $d\phi$ in Lagrangian field theory.
4. The extended phase space has the structure of a fiber bundle where spacetime is the base manifold and the fiber consists of fields and momenta.
5. The allowed field configurations are sections (i.e. mappings from the spacetime into the bundle) which are submanifolds of the bundle characteristic of the 5-form $\Xi = -d\Theta$. This means that tangent vectors of the submanifold annihilate Ξ .
6. Observables of the theory are differential 4-forms integrated over spacetime. For example, $\phi(x)$ is not an observable, but a quantity of type $\int \phi(x)j(x)(*1)$ is, where $j(x)$ is a scalar 'test function' which can be chosen to be non-zero in some small region of spacetime. There is no Poisson bracket in the usual sense because coordinates and momenta are forms of different degrees in this formalism. Instead, we use a generalization of the idea that Poisson bracket between two observables determines rate of change of one observable when the other observable is the Hamiltonian or part of the Hamiltonian. The generalized bracket was defined by Peierls in the early days of quantum field theory.

7. Even in classical mechanics coordinates are zero-forms and momenta are 1-forms. But the base manifold in classical mechanics is one dimensional, and 1-forms in 1-dimension are as good as scalars or 0-forms. Thus coordinates and momenta seem to be on the same footing and the Poisson bracket has a coordinate-momentum symmetry. The Poisson brackets turn out to be equal-time Peierls bracket in classical mechanics as has been shown by the author [2].
8. The symmetries and conserved quantities can be discussed very easily in this formalism. Invariance of the PC-form, expressed as the vanishing Lie derivative $L_Y\Theta = 0$, where Y is the vector field of an infinitesimal symmetry transformation. Noether's theorem holds and the integral of the 3-form $i(Y)\Theta$ over a closed 3-surface is zero. Usually, the three surface is taken made up of two space-like surfaces and a cylindrical surface at infinity. The conservation law takes its usual form in this setting.

The purpose of the present paper is to extend this formalism for gauge fields. In order to preserve gauge invariance we have to modify the structure of PC-form from $*p\wedge d\phi - H_\phi$ to $\Theta_\phi = *p\wedge(d\phi + A\phi) - H_\phi$ where the 1-form A is the gauge connection. In order to include the gauge fields as dynamical fields we must define the PC-form for A as well. Since A is already a 1-form its canonical momentum related linearly to velocities dA would be a 2-form. Calling that F , the PC-form should be $\Theta_A = *F \wedge (dA + A \wedge A) - H_A$ as the natural gauge invariant object.

In the next section we introduce external gauge connection fields A interacting with real scalar fields which have a an $O(N)$ gauge group. We choose scalar fields for simplicity to illustrate the idea. Spinors involve tetrad fields and will be treated separately. In section 3 we develop the formalism for the gauge field alone. The canonical momentum associated to connection field A turns out to be the curvature $dA + A \wedge A$ on allowed characteristic submanifolds. Finally in section 4 interacting fields ϕ and A are considered and equations of motion derived. The issue of gauge fixing and Peierls bracket in this formalism will be treated in a separate communication.

2 Scalar fields in an external gauge field

If there are N real scalar fields then the PC-form for them is

$$\Theta_\phi = *p_i \wedge d\phi^i - H \quad (1)$$

where H is a 4-form depending on all the fields and their momenta. We will omit the index i and write the form simply as $*p \wedge d\phi - H$. If there is an internal space global $O(N)$ rotation (acting on the right) $\phi \rightarrow R^{-1}\phi$ we define the momenta to transform contra-gradiently $p \rightarrow R^T p = pR$. This keeps the PC-form unchanged provided the covariant Hamiltonian H remains unchanged too.

However, if the internal space rotation is ‘gauged’, that is, if matrices R depend on spacetime coordinates t^μ then the PC-form must be modified with the introduction of a gauge connection form A which are new 1-form fields taking values as Lie algebra elements.

We suggest the following expression for the PC-form

$$\Theta_\phi = *p \wedge (d\phi + A\phi) - H(\phi, p). \quad (2)$$

Here A should be regarded as a matrix valued 1-form

$$A = A^a \tau_a = A_\mu^a dt^\mu \tau_a. \quad (3)$$

The matrices τ_a are ‘generators’ (that is Lie algebra basis elements) of the internal symmetry group in the (real) representation to which fields ϕ_i belong. We assume

$$[\tau_a, \tau_b] = C_{ab}^c \tau_c \quad (4)$$

where C_{ab}^c are structure constants of the Lie group. The Lie algebra in the *adjoint* representation, which we need later, is defined by

$$(T_a)_b^c = -C_{ab}^c. \quad (5)$$

It satisfies

$$\text{Tr}[T_a T_b] = k_{ab} \quad (6)$$

where constants k_{ab} allow us to define an inner product on the Lie algebra. It is well-known that k_{ab} is non-degenerate for semi-simple

Lie groups and moreover it is negative definite (as here) for compact groups. We call its inverse by k^{ab} where $k_{ab}k^{bc} = \delta_a^c$.

It clear that the PC-form is invariant if along with $\phi \rightarrow R^{-1}\phi$, the connection transforms as

$$A \rightarrow R^{-1}AR + R^{-1}dR. \quad (7)$$

The form $-d\Theta$ is

$$\begin{aligned} -d\Theta &= -(d * p) \wedge (d\phi + A\phi) + (*p) \wedge d(A\phi) + dH_\phi \\ &= -(d * p) \wedge (d\phi + A\phi) + (*p) \wedge [dA\phi - A \wedge d\phi] + dH_\phi \\ &= -(d * p) \wedge (d\phi + A\phi) - (*p) \wedge A \wedge d\phi + dH_\phi \\ &= -(d * p) \wedge (d\phi + A\phi) - (*p) \wedge A \wedge (d\phi + A\phi) + dH_\phi \\ &= -(d * p + *p \wedge A) \wedge (d\phi + A\phi) + dH_\phi \end{aligned} \quad (8)$$

where in the third step *we have omitted the term $(*p) \wedge dA\phi$ because it has five factors of dt^μ , three for $*p$ and two for*

$$dA = A_{\mu,\nu}^a dt^\nu \wedge dt^\mu \tau_a.$$

Here the connection field A is supposed to be given as an explicit function of spacetime coordinates. Later, when we involve A as a dynamical field we would be required to keep this term because then

$$dA = dA_\mu^a \wedge dt^\mu \tau_a$$

would have only one factor of dt^μ .

For a Hamiltonian of the form

$$H_\phi = \frac{1}{2}(*p) \wedge p + \frac{1}{2}m^2\phi^T\phi(*1) \quad (9)$$

the exterior derivative is

$$\begin{aligned} dH_\phi &= (d * p) \wedge p + m^2\phi(*1) \wedge d\phi \\ &= (d * p) \wedge p + m^2\phi(*1) \wedge (d\phi + A\phi - p) \\ &= (d * p + *p \wedge A) \wedge p + m^2\phi(*1) \wedge (d\phi + A\phi - p) \end{aligned} \quad (10)$$

where we can harmlessly add $A\phi - p$ to $d\phi$ because these terms are 1-forms proportional to dt which when multiplied to $(*1)$ gives zero because of five dt factors. Similarly, keeping an eye on eqn. (8), we can add $*p \wedge A \wedge p = 0$ in the last step above because it has five factors

of type dt^μ which are zero in 4-dimensional spacetime. Substituting this expression for dH_ϕ into (8) we obtain

$$\Xi = -d\Theta = -(d * p + *p \wedge A - m^2 \phi(*1)) \wedge (d\phi + A\phi - p) \quad (11)$$

The factorization of $\Xi = -d\Theta$ allows us a quick route to the field equations. The second factor gives the definition of canonical momentum in terms of velocities

$$p = d\phi + A\phi, \quad (12)$$

and the first factor has terms

$$\begin{aligned} d * p + *p \wedge A &= d * (d\phi + A\phi) + *(d\phi + A\phi) \wedge A \\ &= d * (d\phi + A\phi) - A \wedge *(d\phi + A\phi) \\ &= (d - A) \wedge *(d + A)\phi. \end{aligned} \quad (13)$$

We have a slight abuse of notation here, the term $d \wedge (\dots)$ is just $d(\dots)$. This leads to the equation of motion for the field

$$[(d - A) \wedge *(d + A) - m^2]\phi = 0 \quad (14)$$

3 PC-form for the gauge field alone

In the analysis of the last section the connection field is given externally. To include the field into dynamics, we must write appropriate terms for it in the PC-form. We first write the PC-form just for the connection field.

The connection potential A is already a 1-form matrix which depends on which representation we are considering. In the last section the fields A were in the representation provided by matrices τ_a suitable for acting on the matter fields whereas in this section matrices are in the adjoint representation $(T_a)_b^c = -C_{ab}^c$ for the generators. Strictly, we should call the gauge potential in the two cases by different symbols. But we use the same symbol for simplicity hoping this to be kept in mind.

To express the first term of the fundamental 4-form Θ_A with structure $*(\dots) \wedge dA$ we must define the (star operated) canonical momentum, call it $*F$, a 2-form matrix because dA is already a 2-form. Therefore F is a 2-form matrix as well.

The gauge transformations of A is well-known $A \rightarrow R^{-1}AR + R^{-1}dR$. R is the group matrix in the adjoint representation here.

We also know that

$$\begin{aligned}
dA + A \wedge A &\rightarrow d(R^{-1}AR + R^{-1}dR) \\
&\quad + (R^{-1}AR + R^{-1}dR) \wedge (R^{-1}AR + R^{-1}dR) \\
&= (dR^{-1}) \wedge AR + R^{-1}(dA)R - R^{-1}A \wedge dR \\
&\quad (dR^{-1}) \wedge (dR) + R^{-1}A \wedge AR + R^{-1}A \wedge (dR) \\
&\quad - (dR^{-1}) \wedge AR - (dR^{-1}) \wedge (dR) \\
&= R^{-1}(dA + A \wedge A)R
\end{aligned}$$

where we have used $R^{-1}dR + (dR^{-1})R = d(1) = 0$ above. Thus the PC-form for gauge connection field A should look like

$$\Theta_A = \text{Tr}[*F \wedge (dA + A \wedge A)] - H_A. \quad (15)$$

where F has the transformation $F \rightarrow R^{-1}FR$.

The Hamiltonian 4-form is taken to be

$$H_A = \frac{1}{2} \text{Tr}(*F) \wedge F \quad (16)$$

which gives

$$\begin{aligned}
\Xi &= -d\Theta_A \\
&= -d[\text{Tr} * F \wedge (dA + A \wedge A)] + dH_A \\
&= -\text{Tr}[(d * F) \wedge (dA + A \wedge A)] - \text{Tr}[*F \wedge (dA \wedge A - A \wedge dA)] \\
&\quad + \text{Tr}(d * F) \wedge F
\end{aligned}$$

where we have used the formula

$$d(*F \wedge F) = 2(d * F) \wedge F.$$

Now, we can write

$$\text{Tr}[*F \wedge dA \wedge A] = \text{Tr}[A \wedge *F \wedge dA] = \text{Tr}[A \wedge *F \wedge (dA + A \wedge A - F)]$$

where we have added two terms which are actually zero because they carry five factors of dt . Similarly,

$$\text{Tr}[*F \wedge A \wedge dA] = \text{Tr}[*F \wedge A \wedge (dA + A \wedge A - F)].$$

Thus

$$-d\Theta_A = -\text{Tr}[(d * F + A \wedge *F - *F \wedge A) \wedge (dA + A \wedge A - F)]$$

The equations of motion are finally,

$$d * F + A \wedge *F - *F \wedge A = 0, \quad F = dA + A \wedge A \quad (17)$$

4 Matter and gauge fields together

When both the matter fields ϕ and gauge fields A are present, the PC-form is the sum of the two

$$\Theta = \Theta_\phi - H_\phi + \Theta_A - H_A. \quad (18)$$

We vary both ϕ and A fields. The expressions have been calculated before and the only change now is that the term $*p \wedge dA\phi$ which we threw away in $-d\Theta_\phi$ when A was an external field depending explicitly on spacetime, has to be kept and combined with the similar dA term in $-d\Theta_A$. There is also a minor problem of changing A in the τ_a representation to the adjoint representation.

The concerned term is $(*p) \wedge dA^a \tau_a \phi$. From $A = A^a T_a$ we can take the trace and infer $\text{Tr}(AT_b) = A^a k_{ab}$. Therefore, using the inverse matrix, $A^a = k^{ab} \text{Tr}(AT_b)$. The combined PC-form is then

$$\begin{aligned} \Xi = & -(d*p + *p \wedge A - m^2 \phi(*1)) \wedge (d\phi + A\phi - p) \\ & + (*p) \tau_a \phi k^{ab} \text{Tr}[T_b(dA + A \wedge A)] \\ & - \text{Tr}[(d*F + A \wedge *F - *F \wedge A) \wedge (dA + A \wedge A - F)] \end{aligned}$$

where we have completed dA to $dA + A \wedge A - F$ because $A \wedge A - F$ has two factors of dt which will give zero when multiplied to three in $*p$. Thus finally we can bring this term in the last set of terms

$$\begin{aligned} \Xi = & -(d*p + *p \wedge A - m^2 \phi(*1)) \wedge (d\phi + A\phi - p) \\ & - \text{Tr}[(d*F + A \wedge *F - *F \wedge A - (*p \tau_a \phi) k^{ab} T_b) \\ & \wedge (dA + A \wedge A - F)] \end{aligned} \quad (19)$$

The Hamiltonian equations are therefore as follows. The definition of momenta is given by

$$p = d\phi + A\phi, \quad (20)$$

$$F = dA + A \wedge A, \quad (21)$$

and the ‘field equations’ are, finally,

$$d*p + *p \wedge A - m^2 \phi(*1) = 0, \quad (22)$$

$$d*F + A \wedge *F - *F \wedge A - (*p \tau_a \phi) k^{ab} T_b = 0. \quad (23)$$

A Notation

We use notation as given for example in [3] or [4].

The spacetime is a Riemannian space with coordinates $t^\mu, \mu = 0, 1, 2, 3$. Basis vectors in a tangent space are written $\partial_\mu = \partial/\partial t^\mu$. The metric is given by the inner product $\langle \partial_\mu, \partial_\nu \rangle = g_{\mu\nu}$. The cotangent spaces have basis elements dt^μ with $\langle dt^\mu, dt^\nu \rangle = g^{\mu\nu}$. The metric has signature $(-1, 1, 1, 1)$. The wedge product is defined so that $\alpha \wedge \beta = \alpha \otimes \beta - \beta \otimes \alpha$ for one-forms α and β . The exterior derivative is defined so that for an r -form $\alpha = a_{\mu_1 \dots \mu_r} dt^{\mu_1} \wedge \dots \wedge dt^{\mu_r}$ the derivative is the $(r+1)$ -form

$$d\alpha = a_{\mu_1 \dots \mu_r, \nu} dt^\nu \wedge dt^{\mu_1} \wedge \dots \wedge dt^{\mu_r}.$$

The Hodge star is a linear operator that maps r -forms into $(4-r)$ -forms in our four-dimensional space. The definition is

$$\begin{aligned} *(dt^{\mu_1} \wedge \dots \wedge dt^{\mu_r}) &= [(4-r)!]^{-1} \sqrt{-g} g^{\mu_1 \nu_1} \dots \\ &\quad g^{\mu_r \nu_r} \varepsilon_{\nu_1 \dots \nu_r \nu_{r+1} \dots \nu_4} dt^{\nu_{r+1}} \wedge \dots \wedge dt^{\nu_4} \end{aligned}$$

where g denotes the determinant of $g_{\mu\nu}$ and ε is the antisymmetric tensor defined with $\varepsilon_{0123} = 1$. The one-dimensional space of 0-forms has the unit vector equal to real number 1. The one-dimensional space of 4-forms has the chosen orientation given by the unit vector $\varepsilon = n^0 \wedge n^1 \wedge n^2 \wedge n^3$ where n^μ are the orthonormal basis vectors in the four-dimensional space of 1-forms. In the coordinate basis $\varepsilon = \sqrt{-g} dt^0 \wedge dt^1 \wedge dt^2 \wedge dt^3$. The star operator acting on the zero form equal to constant number 1 is denoted by $*1 = \varepsilon = \sqrt{g} dt^0 \wedge dt^1 \wedge dt^2 \wedge dt^3$. We have the simple result that $dt^\mu \wedge *dt^\nu = -*dt^\nu \wedge dt^\mu = g^{\mu\nu}(*1)$.

The interior product $i(X)$ of a vector X with an r -form α gives an $(r-1)$ -form $i(X)\alpha$ defined by

$$(i(X)\alpha)(Y_1, \dots, Y_{r-1}) = \alpha(X, Y_1, \dots, Y_{r-1})$$

When it is more convenient we will denote the interior product operator by i_X in place of $i(X)$.

Two successive applications of interior products on a form are denoted by

$$i(X, Y)\alpha \equiv [i(X) \circ i(Y)]\alpha = i(X)[i(Y)\alpha]$$

Note that $i(X, Y) = -i(Y, X)$. Similarly successive applications $i(XY \dots Z)$ of many such interior products can be defined. If α is an

r -form then

$$i(X)(\alpha \wedge \beta) = [i(X)\alpha] \wedge \beta + (-1)^r \alpha \wedge i(X)\beta$$

References

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